Inner functions, real Hilbert subspaces and new Boundary QFT nets of von Neumann algebras

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Mainly based on papers with K.H. Rehren and a joint work with E. Witten
Things to discuss

- Inner functions and Beurling-Lax theorem
- Real Hilbert subspaces
- Algebraic Boundary Conformal Field Theory (K.H. Rehren, R.L.)
- Models of Boundary QFT (E. Witten, R.L.)
Inner functions

\[ \mathbb{D} \equiv \{ z \in \mathbb{C} : |z| < 1 \} \text{ unit disk, } \mathbb{H}^\infty(\mathbb{D}) \text{ Hardy space.} \]

\[ \varphi \in \mathbb{H}^\infty(\mathbb{D}) \Rightarrow \exists \varphi(e^{i\theta}) \equiv \lim_{r \to 1^-} \varphi(re^{i\theta}) \text{ a.e. on } \partial\mathbb{D} \]

\[ \varphi \in \mathbb{H}^\infty(\mathbb{D}) \text{ is an inner function if } |\varphi(z)| = 1 \text{ for almost all } z \in \partial\mathbb{D}. \]

Examples:

\[ B_0(z) \equiv z, \text{ or its Möbius transform: } \]

\[ B_a(z) = \frac{|a|}{a} \frac{z-a}{1-\overline{a}z} \text{ (Blaschke factor),} \]

\[ B(z) \equiv \prod_{n=1}^{\infty} B_{a_n}(z) \text{ (Blaschke product),} \]

\[ a_n \in \mathbb{D}, \sum_{n=1}^{\infty}(1 - |a_n|) < \infty. \]
$B(z)$ has zeros exactly at $\{a_n\}$, with multiplicity.
If an inner function $\varphi$ has no zeros on $\mathbb{D}$, then $\varphi$ is called a
\textit{singular} inner function.
$\varphi$ is an inner function iff (uniquely)

$$
\varphi(z) = \alpha B(z) \exp \left( - \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(e^{i\theta}) \right),
$$

where $\mu$ is a positive, Lebesgue singular measure on $\partial \mathbb{D}$, $B(z)$ is a
Blaschke product and $\alpha$ is a constant with $|\alpha| = 1$. All the zeros of
$\varphi$ come from $B$ so $\varphi$ is singular if and only if $B$ is the identity.
Inner functions form a (multiplicative) \textit{semigroup}, singular inner
functions a sub-semigroup.

One-param. semigroup $\{\varphi_t\}$ of inner functions:

$$
\varphi_t(z) = e^{it\lambda} \exp \left( -t \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(e^{i\theta}) \right)
$$
Symmetric inner functions:

$\varphi$ is symmetric $\bar{\varphi}(z) = \varphi(\bar{z})$.

Notions go $\mathbb{S}_\infty$ and $\mathbb{S}_\pi$: $h(z) \equiv i \frac{1 + z}{1 - z}$,

$$\mathbb{D} \xrightarrow{h} \mathbb{S}_\infty \xrightarrow{\log} \mathbb{S}_\pi$$

$\varphi \in \mathbb{H}_\infty(\mathbb{S}_\pi)$ inner: $|\varphi(q)| = |\varphi(q + i\pi)| = 1$

symmetric: $\varphi(q + i\pi) = \bar{\varphi}(q)$, $q \in \mathbb{R}$ a.e.

$\varphi \in \mathbb{H}^\infty(\mathbb{S}_\infty)$ inner: $|\varphi(q)| = 1$, $q > 0$

symmetric: $\varphi(-q) = \bar{\varphi}(q)$ a.e.

Scattering functions

A scattering function is a symmetric innere function $f$ on $\mathbb{S}_\pi$ s.t. $\varphi(-p) = \varphi(p)$.

Inverse scattering program: construct QFT models from scattering function.
Beurling-Lax theorem (1949-1959)

$S$ shift operator on $H^2(\mathbb{D})$:

$$Sf(z) = zf(z)$$

A closed $S$-invariant subspace $K$ of $H^2(\mathbb{D})$ has the form

$$K = \varphi H^2(\mathbb{D}), \quad \varphi \text{ an inner function}$$

$\downarrow$

$f \in H^2$ (or $f \in H^p$, $p \geq 1$) has a factorization

$$f(z) = \varphi(z)\psi(z)$$

$\varphi$ is inner and $\psi$ is outer

$$\psi(z) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})| \, dt\right)$$

Lax generalization to $H^2(\mathbb{S}_\infty)$, one-param. unitary translations.
Standard real Hilbert subspaces

\( \mathcal{H} \) complex Hilbert space and \( H \subset \mathcal{H} \) a real linear subspace.

Symplectic complement:

\[ H' \equiv \{ \xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \ \forall \eta \in H \}. \]

\( H' = (iH)^\perp \) (real orthogonal complement)

\[ H_1 \subset H_2 \Rightarrow H_1' \supset H_2'. \]

A standard subspace \( H \) of \( \mathcal{H} \) is a closed, real linear subspace of \( \mathcal{H} \) which is both cyclic (\( H + iH = \mathcal{H} \)) and separating (\( H \cap iH = \{0\} \)). \( H \) is standard iff \( H' \) is standard.

\( H \) standard subspace \( \rightarrow \) anti-linear operator

\[ S \equiv S_H : D(S) \subset \mathcal{H} \rightarrow \mathcal{H}, \text{ where } D(S) \equiv H + iH, \]

\[ S : \xi + i\eta \mapsto \xi - i\eta , \quad \xi, \eta \in H. \]

\[ S^2 = 1 \restriction_{D(S)}. \] \( S \) is closed and densely defined.
Conversely, $S$ densely defined, closed, anti-linear involution on $\mathcal{H}$ gives

$$H = \{\xi : S\xi = \xi\}$$ is a standard subspace

$$H \longleftrightarrow S$$ bijection

**Modular theory.** Set

$$S_H = J_H \Delta_H^{1/2}$$

polar decomposition of $S = S_H$. Then $J_H$ is an anti-unitary involution $\Delta \equiv S^*S > 0$

$$\Delta_H^{-it} H = H, \quad J_H H = H'$$

**Borchers theorem (real subspace version)**

$H$ standard subspace, $U$ a one-parameter group with positive generator

$$U(s)H \subset H \quad s \geq 0.$$ Then:

$$\begin{cases}
\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t}s), \\
J_H U(s) J_H = U(-s), \quad t, s \in \mathbb{R}.
\end{cases}$$
**Note:** Setting \( K \equiv U(1)H \) we have

\[
\Delta_{H}^{-it}K = \Delta_{H}^{-it}U(1)H = U(e^{2\pi t})\Delta_{H}^{-it}H \\
= U(e^{2\pi t})H \subset K, \quad t \geq 0.
\]

\( K \subset H \) is a half-sided modular inclusion.

**About the proof (adapted from Florig).** With \( \xi \in H, \xi' \in H' \)

\[
f_U(z) = (\Delta^{iz} \xi', U(e^{2\pi z} s) \Delta^{-iz} \xi).
\]

is analytic in \( S_{1/2} = \{ z \in \mathbb{C} : 0 < \Im z < \frac{1}{2} \} \) (the generator of \( U(t) \) is positive and \( \Im e^{2\pi z} s \geq 0 \) for \( z \in S_{1/2} \)).

\[
V(t) = JU(-t)J \text{ satisfies the same assumptions then } U \text{ because of } JH = H' \\
f_U \left( t + \frac{i}{2} \right) = (\Delta^{-1/2} \Delta^{-it} \xi, U(e^{2\pi t + i s} \Delta^{-it} \Delta^{1/2} \xi) \\
= (\Delta^{-1/2} \Delta^{-it} \xi, JV(e^{2\pi t} s) \Delta^{-it} \xi) \\
= (\Delta^{-it} \xi, (J \Delta^{1/2}) V(e^{2\pi t} s) \Delta^{-it} \xi) \\
= (\Delta^{-it} \xi, V(e^{2\pi t} s) \Delta^{-it} \xi) = f_V(t)
\]
(KMS and positivity of energy) analogously

\[ V(t) = JU(-t)J \] satisfies the same assumptions then \( U \) because of \( JH = H' \)

\[ f_V \left( t + \frac{i}{2} \right) = f_U(t) \]

\( f_U \) and \( f_V \) glue to an entire bounded function, thus constant.

Converse: Wiesbrock, Borchers, Araki-Zsido theorem (real subspace version)

Let \( H, K \) be standard subspaces. Assume half-sided modular inclusion:

\[ \Delta^{-it}_H K \subset K, \quad t \geq 0 \]

Then \( \{ \Delta^{it}_K, \Delta^{is}_H \} \) generates a unitary representation of the "ax+b" group with positive energy
dilation group = \( \Delta^{-is/2\pi}_H \)
gen. of translations \( P = \frac{1}{2\pi} (\log \Delta_K - \log \Delta_H) \)
Möbius covariant nets of real Hilbert subspaces

A local Möbius covariant net of standard subspaces $\mathcal{A}$ of real Hilbert subspaces on the intervals of $S^1$ is a map

$$I \rightarrow H(I)$$

with

1. Isotony: If $I_1$, $I_2$ are intervals and $I_1 \subset I_2$, then

$$H(I_1) \subset H(I_2).$$

2. Möbius invariance: There is a unitary representation $U$ of Mob on $\mathcal{H}$ such that

$$U(g)H(I) = H(gI), \quad g \in \text{Mob}, \ I \in \mathcal{I}.$$ 

Here $\text{Mob} \simeq PSL(2, \mathbb{R})$ acts on $S^1$ as usual.
3. Positivity of the energy: $L_0 \geq 0$

4. Cyclicity: the complex linear span of all spaces $H(I)$ is dense in $\mathcal{H}$.

5. Locality: If $I_1$ and $I_2$ are disjoint intervals then

$$H(I_1) \subset H(I_2)'$$

First consequences

Irreducibility: $\overline{\text{real lin.span}_{I \in \mathcal{I}} H(I)} = H$.

Reeh-Schlieder theorem: $H(I)$ is a standard subspace for every $I$.

Bisognano-Wichmann property: Tomita-Takesaki modular operator $\Delta_I$ and conjugation $J_I$ of
\( H(I) \), are

\[
U(\Lambda_I(2\pi t)) = \Delta_I^{-it}, \quad t \in \mathbb{R}, \quad \text{dilations}
\]

\[
U(r_I) = J_I \quad \text{reflection}
\]

\[
(\Lambda_{I_1}(t)x = e^{tx}, x \in \mathbb{R}, \quad I_1 \sim \mathbb{R}^+ \quad \text{upper semi-circle})
\]

\textbf{Haag duality:} \( H(I)' = H(I') \quad (I' \equiv S^1 \setminus I) \).

\textbf{Factoriality:} \( H(I) \cap H(I)' = 0 \)

\textbf{Additivity:} \( I \subset \bigcup_i I_i \implies H(I) \subset \overline{\text{real lin.span}_i H(I_i)} \).

\textbf{Modular theory and representations of } SL(2, \mathbb{R})
(Brunetti, Guido, L.)

\( U \) a unitary, positive energy representation of \( \text{Mob} \) on \( \mathcal{H} \) and \( J \) anti-unitary involution on \( \mathcal{H} \) s.t.

\[
JU(g)J = U(rgr), \quad g \in \text{Mob}
\]
where $r : z \mapsto \bar{z}$ reflection on $S^1$ w.r.t. the upper semicircle $I_1$. Then define

$$J_I \equiv U(g)JU(g)^*$$

where $g \in \text{Mob}$ maps $I_1$ onto $I$.

$$\Delta^{it}_I \equiv U(\Lambda_I(-2\pi t)), \quad t \in \mathbb{R}$$

namely $-\frac{1}{2\pi} \log \Delta_I$ generator of dilations of $I$,

$$S_I \equiv J_I \Delta^{1/2}_I$$

is a densely defined, antilinear, closed involution on $\mathcal{H}$.

$H(I)$ standard subspace associated with $S_I$}

↓

Möbius covariant local net of real Hilbert spaces

A ±hsm factorization of real subspaces is a triple $K_0, K_1, K_2$, where $\{K_i, i \in \mathbb{Z}_3\}$ is a set of
standard subspaces s.t. $K_i \subset K'_{i+1}$ is a ±hsm inclusion.

Factorization

\[ \uparrow \]

Local Möbius covariant net of real Hilbert spaces

\[ \uparrow \]

Positive energy representation of $SL(2, \mathbb{R})/\{1, -1\}$

Note: Irr. positive energy rep. of $SL(2, \mathbb{R})/\{1, -1\}$ are parametrized by $\mathbb{N}$
Endomorphisms of standard subspaces

A standard pair of $\mathcal{H}$ is a pair $(H, T)$ such that

- $H$ is a standard subspace,
- $T$ is a one-par. unitary group, with positive generator $P$, s.t. $T(t)H \subset H, \ t \geq 0$.

**Thm.** Assume $(H, T)$ to be irreducible and let $K \subset H$ be a real subspace. The following are equivalent:

(i) $T(t)K \subset K, \ t \geq 0$,

(ii) $K = VH$ where $V$ is a unitary commuting with $T$,

(iii) $K = VH$ where $V = \psi(Q)$ with $Q \equiv \log P$ and $\psi \in L^\infty(\mathbb{R}, dq)$ is the boundary value of an inner function in $H^\infty(\mathbb{S}_\pi)$ such that $\psi(q + i\pi) = \overline{\psi}(q)$, for almost all $q \in \mathbb{R}$.

The semigroup $\mathcal{E}(H)$ of endomorphisms of $(H, T)$ is isomorphic to the semigroup of symmetric inner functions on the strip $0 < \Im z < \pi$. 
2-dimensional CFT

\[ M = \mathbb{R}^2 \] Minkowski plane.

\[
\begin{pmatrix}
    T_{00} & T_{10} \\
    T_{01} & T_{11}
\end{pmatrix}
\] conserved and traceless stress-energy tensor.

As is well known, \( T_L = \frac{1}{2}(T_{00} + T_{01}) \) and \( T_R = \frac{1}{2}(T_{00} - T_{01}) \) are chiral fields,

\[
T_L = T_L(t + x), \quad T_R = T_R(t - x).
\]

Left and right movers.
Two-dimensional conformal fields and nets

\( \Psi_k \) family of conformal fields on \( M \): \( T_{ij} + \text{relatively local fields} \)
\( \mathcal{O} = I \times J \) double cone, \( I, J \) intervals of the chiral lines \( t \pm x = 0 \)

\[ \mathcal{A}(\mathcal{O}) = \{ e^{i \Psi_k(f)}, \text{supp} f \subset \mathcal{O} \}'' \]

then by relative locality

\[ \mathcal{A}(\mathcal{O}) \supset \mathcal{A}_L(I) \otimes \mathcal{A}_R(J) \]

\( \mathcal{A}_L, \mathcal{A}_R \) chiral fields on \( t \pm x = 0 \) generated by \( T_L, T_R \) and other chiral fields

(completely) rational case: \( \mathcal{A}_L(I) \otimes \mathcal{A}_R(J) \subset \mathcal{A}(\mathcal{O}) \) finite Jones index
Local conformal nets

A local Möbius covariant net $\mathcal{A}$ on $S^1$ is a map

$$I \in \mathcal{I} \rightarrow \mathcal{A}(I) \subset B(\mathcal{H})$$

$I \equiv$ family of proper intervals of $S^1$, that satisfies:

- **A. Isotony.** $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B. Locality.** $I_1 \cap I_2 = \emptyset \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C. Möbius covariance.** $\exists$ unitary rep. $U$ of the Möbius group $\text{Möb}$ on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Möb}, \quad I \in \mathcal{I}.$$

- **D. Positivity of the energy.** Generator $L_0$ of rotation subgroup of $U$ (conformal Hamiltonian) is positive.
- **E. Existence of the vacuum.** $\exists!$ $U$-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and $\Omega$ is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$. 
First consequences

- **Irreducibility**: \( \bigvee_{I \in \mathcal{I}} A(I) = B(H) \).
- **Reeh-Schlieder theorem**: \( \Omega \) is cyclic and separating for each \( A(I) \).
- **Bisognano-Wichmann property**: Tomita-Takesaki modular operator \( \Delta_I \) and conjugation \( J_I \) of \( (A(I), \Omega) \), are
  \[
  U(\Lambda_I(2\pi t)) = \Delta_I^{it}, \quad t \in \mathbb{R}, \quad \text{dilations}
  
  U(r_I) = J_I \quad \text{reflection}
  \]
  (Frölich-Gabbiani, Guido-L.)
- **Haag duality**: \( \mathcal{A}(I)' = \mathcal{A}(I') \)
- **Factoriality**: \( \mathcal{A}(I) \) is \( \text{III}_1 \)-factor (in Connes classification)
- **Additivity**: \( I \subset \bigcup_i I_i \implies \mathcal{A}(I) \subset \bigvee_i \mathcal{A}(I_i) \) (Fredenhagen, Jorss).
Representations

A representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map

$$l \in \mathcal{I} \mapsto \pi_l, \text{ normal rep. of } \mathcal{A}(l) \text{ on } B(\mathcal{H})$$

$$\pi_{\tilde{l}}|_{\mathcal{A}(l)} = \pi_l, \quad l \subset \tilde{l}$$

$\pi$ is automatically diffeomorphism covariant: $\exists$ a projective, pos. energy, unitary rep. $U_\pi$ of $\text{Diff}^{(\infty)}(S^1)$ s.t.

$$\pi_{g\tilde{l}}(U(g)xU(g)^*) = U_\pi(g)\pi_l(x)U_\pi(g)^*$$

for all $l \in \mathcal{I}$, $x \in \mathcal{A}(l)$, $g \in \text{Diff}^{(\infty)}(S^1)$ (Carpi & Weiner)

**DHR argument:** given $l$, there is an endomorphism of $\mathcal{A}$ localized in $l$ equivalent to $\pi$; namely $\rho$ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to $\pi$, such that $\rho_{l'} = \text{id} \upharpoonright_{\mathcal{A}(l')}$.  

- $\text{Rep}(\mathcal{A})$ is a *braided tensor category* (DHR, FRS, L.)
A satisfies the *split* property if the von Neumann algebra

\[ A(I_1) \lor A(I_2) \cong A(I_1) \otimes A(I_2) \]

(natural isomorphism) if \( \overline{I}_1 \cap \overline{I}_2 = \emptyset \).

\[ \text{Tr}(e^{-tL_0}) < \infty, \ \forall t > 0 \implies \text{split} \]
Complete rationality

\[ l_1, l_2 \text{ intervals } \overline{l_1} \cap \overline{l_2} = \emptyset, \ E \equiv l_1 \cup l_2. \]

\[ \mu\text{-index}: \ \mu_{\mathcal{A}} \equiv [\mathcal{A}(E')': \mathcal{A}(E)] \]

(Jones index). \ \mathcal{A} \text{ conformal:}

\[ \mathcal{A} \text{ completely rational} \overset{\text{def}}{=} \mathcal{A} \text{ split} & \mu_{\mathcal{A}} < \infty \]

**Thm.** (Y. Kawahigashi, M. Müger, R.L.) \( \mathcal{A} \) completely rational: then

\[ \mu_{\mathcal{A}} = \sum_i d(\rho_i)^2 \]

sum over all irreducible sectors. (F. Xu in \( SU(N) \) models);

- \( \mathcal{A}(E) \subset \mathcal{A}(E')' \sim \text{LR inclusion (quantum double)}; \)
- Representations form a modular tensor category (i.e. non-degenerate braiding).
Boundary CFT

Stress-energy tensor left/right movers $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$: $T_L = T_L(t + x)$, $T_R = T_R(t - x)$.

Boundary condition: no energy flow across the boundary:

$$T_{01}(t, x = 0) = 0 \iff T_L = T_R \equiv T.$$ 

so $T_{10} = T_{01}, T_{11} = T_{00}$ are of the form

$T_{00}(t, x) = T(t + x) + T(t - x), \quad T_{01}(t, x) = T(t + x) - T(t - x),$

i.e., bi-local expressions in terms of $T$.
The chiral fields of a boundary CFT generate a net

\[ O \mapsto A_+(O). \]

\( A_+(O) \) is generated by chiral fields smeared in the variable \( t + x \) over the interval \( I \) and in the variable \( t - x \) over the interval \( J \), where \( O = I \times J, I > J \), is an open double-cone in \( M_+ \). The bi-localized structure translates into the form of the local algebras

\[ A_+(O) = A(I) \lor A(J) \quad (O = I \times J, \ I > J). \]
Definition of Boundary CFT

A *boundary CFT (BCFT)* associated with $A$ is a local, isotonous net $O \mapsto B_+(O)$ over the double-cones within the half-space $M_+$, represented on a Hilbert space $\mathcal{H}_B$ such that

(i) there is a unitary representation $\mathcal{U}$ of the covering of the Möbius group $PSL(2, \mathbb{R})$ with positive generator for the subgroup of translations, such that

$$\mathcal{U}(g)B_+(O)\mathcal{U}(g)^* = B_+(gO)$$

(ii) There is a representation $\pi$ of $A$ on $\mathcal{H}_B$ such that $B_+(O)$ contains $\pi(A_+(O))$ and $\pi$ is $\mathcal{U}$-covariant.

(iii) *“Joint irreducibility”*: For each double-cone $O$, $B_+(O) \vee \pi(A_+)$ is irreducible on $\mathcal{H}_B$ (almost automatic)
If $I \mapsto B(I)$ is an irreducible chiral extension of $I \mapsto A(I)$ (possibly non-local, but relatively local with respect to $A$), then the \textit{induced net} is defined by

$$O \mapsto B_{+}^{ind}(O) := B(L) \cap B(K)'.$$
BCFT $\rightarrow$ non-local chiral net

A boundary CFT $O \mapsto B_+(O)$ generates a chiral net $I \mapsto B^{gen}(I)$ (the associated boundary net) on $\mathcal{H}_B$, by

$$B^{gen}(I) := \bigvee_{O \subset W_L} B_+(O) \equiv B_+(W_L)$$

where $W_L$ is the left wedge spanned by $I$

The observables of the associated chiral boundary net localized in $I$ are generated by BCFT observables localized in double cones.
(i) In the special case $B = A$, the induced net is the dual net $A_{+}^{dual}$:

$$B_{dual}^{dual}(O) \equiv B(O')'$$

so $A_{+}(O) \subset A_{+}^{dual}(O)$ is the 2-interval inclusion.

(ii) If $B$ is a chiral extension of $A$, then

$$(B_{+}^{ind})^{gen} = B$$

Conversely

$$(B_{+}^{gen})^{ind} = B_{+}^{dual}$$

(iii) Every induced net $B_{ind_{+}}$ is self-dual (Haag dual).

conclusion:

non-local chiral extensions of $A \leftrightarrow$ local extensions of $A_{+}$

Classification for $c < 1$: Kawahigashi, Pennig, Rehren, L.
Remarkable properties

Let $B$ chiral extension of $A$, and $B \text{ind}_+$ the induced BCFT net.

(i) The index of $\pi(A_+(O)) \subset B \text{ind}_+(O)$ equals the $\mu$-index $\mu_A$ of $A$. This index is thus the same for each chiral extension

(ii) When $B_+$ is Haag dual, then $\mu_{B_+} = 1$, and $B_+$ satisfies Haag duality also for the disconnected regions of the form $E = O_1 \cup O_2$

(iii) A Haag dual boundary CFT net $B_+$ has the no nontrivial DHR sectors.
The semigroup $\mathcal{E}(\mathcal{A})$

Let $\mathcal{A}$ be a local Möbius covariant net of von Neumann algebras on $\mathbb{R}$

$$I \subset \mathbb{R} \text{ interval } \to \mathcal{A}(I)$$

$T$ one-parameter unitary translation group. Then

$T(t)\mathcal{A}(I)T(-t) = \mathcal{A}(I + t)$, $T$ has positive generator $P$ and

$T(t)\Omega = \Omega$ where $\Omega$ is the vacuum vector.

Let $V$ be a unitary on $\mathcal{H}$ commuting with $T$. The following are equivalent:

$(i)$ $V\mathcal{A}(I_2)V^* \text{ commutes with } \mathcal{A}(I_1)$ for all intervals $I_1, I_2$ of $\mathbb{R}$ such that $I_2 > I_1$ ($I_2$ is contained in the future of $I_1$).

$(ii)$ $V\mathcal{A}(a, \infty)V^* \subset \mathcal{A}(a, \infty)$ for every $a \in \mathbb{R}$.

$(iii)$ $V\mathcal{A}(0, \infty)V^* \subset \mathcal{A}(0, \infty)$. 
The semigroup $\mathcal{E}(\mathcal{A})$

$\mathcal{E}(\mathcal{A}) \equiv$ semigroup of unitaries $V$ as above

$\mathcal{A}$ conformal net & $V \in \mathcal{E}(\mathcal{A}) \longrightarrow$ Boundary QFT $\mathcal{A}_V$

$\mathcal{A}_V(\mathcal{O}) \equiv \mathcal{A}(I_1) \vee V\mathcal{A}(I_2)V^*$

where $I_1, I_2$ are intervals of time-axis such that $I_2 > I_1$ and $\mathcal{O} = I_1 \times I_2$. 
$\mathcal{A}$ with the split property, $V \in \mathcal{E}(\mathcal{A})$ then $\mathcal{A}_V$ is locally isomorphic to $\mathcal{A}_+ = \mathcal{A}_I$.

As an immediate consequence, if $V_t$ is a one-parameter semigroup of unitaries in $\mathcal{E}(\mathcal{A})$, the family $\mathcal{A}_{V_t}$ gives a deformation of the conformal net $\mathcal{A}_+$ on $M_+$ with translation covariant nets on $M_+$ that are locally isomorphic to $\mathcal{A}_+$. 
Constructing models

A free field on $\mathbb{R}$ acting on the Fock space $F(\mathcal{H})$.

$H$ standard subspace of $\mathcal{H} \rightarrow$ von Neumann algebra on $F(\mathcal{H})$

$$\mathcal{A}(H) = \{ W(h) : h \in H \}''$$

Take $H = H(0, \infty)$.

$$V \in \mathcal{E}(H) \rightarrow \Gamma(V) \in \mathcal{E}(\mathcal{A})$$

therefore

symmetric inner function $\rightarrow$ $V \in \mathcal{E}(\mathcal{A}) \rightarrow$ Boundary QFT net $\mathcal{A}_V$ on $M_+$

In particular

$\varphi$ scattering function $\rightarrow$ Boundary QFT
More general BQFT’s

\[ \mathcal{A} = \mathcal{A}_N \] Buchholz-Mach-Tododrov extension of \( U(1) \)-current net:

symmetric inner function Hölder continuous at 0 & \( V \in \mathcal{E}(\mathcal{A}) \)

\[ \downarrow \]

Boundary QFT net \( \mathcal{A}_V \) on \( \mathcal{M}_+ \)

Examples: \( \mathcal{A}_1 \) associated with level 1 \( \widehat{su}(2) \)-Kac-Moody algebra with \( c = 1 \), \( \mathcal{A}_2 \) Bose subnet of free complex Fermi field net, \( \mathcal{A}_3 \) appears in the \( \mathbb{Z}_4 \)-parafermion current algebra analyzed by Zamolodchikov and Fateev, and in general \( \mathcal{A}_N \) is a coset model \( SO(4N)_1/\text{SO}(2N)_2 \).
Outlook and problems

- Models on the full Minkowski plane
- Which BQFT’s are associated with loop group models?
- Given a completely rational \( \mathcal{A} \) CFT on the boundary, do all BQFT’s \( \mathcal{A}_\mathcal{V} \) have the same positive energy representations?
- Construct BQFT on different spacetimes