Chapter 7

Free-discontinuity Problems

We now consider minimization problems for functionals whose natural domains are sets of functions which admit a finite number of discontinuities. The set of these discontinuities will be an unknown of the problems, and for this reason the latter will be called ‘free-discontinuity problems’.

7.1 Piecewise-Sobolev functions

To have a precise statement of free-discontinuity problems, it will be useful to define some spaces of piecewise weakly-differentiable functions.

Definition 7.1 Let \( 1 \leq p \leq +\infty \). We define the space \( P^{1,p}(a, b) \) of piecewise-W\(^{1,p}\) functions on the bounded interval \( (a, b) \) as the direct sum

\[
P^{1,p}(a, b) = W^{1,p}(a, b) \oplus PC(a, b), \tag{7.1}
\]

i.e. \( u \in P^{1,p}(a, b) \) if and only if \( v \in W^{1,p}(a, b) \) and \( w \in PC(a, b) \) exist such that \( u = v + w \). Note that \( W^{1,p}(a, b) \cap PC(a, b) \) equals the set of constant functions, so that \( u \) and \( v \) are uniquely determined up to an additive constant. The function \( u \) inherits the notation valid for \( v \) and \( w \); namely, we define the jump set of \( u \) and the weak derivative of \( u \) as

\[
S(u) = S(w) \quad \text{and} \quad u' = v', \tag{7.2}
\]

respectively. Moreover, the left and right-hand side values of \( u \) are defined by

\[
u^\pm(x) = v(x) + w^\pm(x). \tag{7.3}
\]

Remark 7.2 Clearly, \( u \in P^{1,p}(a, b) \) if and only if there exist \( a = t_0 < t_1 < \ldots < t_N = b \) such that \( u \in W^{1,p}(t_{i-1}, t_i) \) for all \( i = 1, \ldots, N \). With this definition \( S(u) \) is interpreted as the minimal of such sets of points, and \( u \in L^2(a, b) \) is defined piecewise on \( (a, b) \setminus S(u) \).
7.2 Some model problems

Even though the treatment of minimization problems for functionals defined on $P^1_{-W^p,1}(a,b)$ with $p > 1$ will be easily dealt with by combining the results that we have already proved for functionals defined on Sobolev functions and on piecewise-constant functions we illustrate their importance with two examples.

7.2.1 Signal reconstruction: the Mumford-Shah functional

As for functionals defined on piecewise-constant functions a model for signal reconstruction can be proposed using piecewise-Sobolev functions. Mumford and Shah proposed a model which can be translated in dimension one in the following: Given a datum $g$ (the distorted signal) recover the original piecewise-smooth signal $u$ by solving the problem

$$
\min \left\{ c_1 \int_{(a,b)} |u'|^2 \, dt + c_2 \#(S(u)) + c_3 \int_{(a,b)} |u-g|^2 \, dt : u \in P-W^{1,2}(a,b) \right\}. \quad (7.4)
$$

The parameters $c_1, c_2, c_3$ are tuning parameters. A large $c_1$ penalizes high gradients (in a sense, we can regard the corresponding segmentation problem in Chapter 2, where functions with non-zero gradients are not allowed, as that in (2.1) with $c_1 = +\infty$), a large $c_2$ forbids the introduction of too many discontinuity points, and $c_3$ controls the distance of $u$ to $g$.

7.2.2 Fracture mechanics: the Griffith functional

A simple approach to some problems in the mechanics of brittle solids is that proposed by Griffith, which can be stated more or less like this: Each time a crack is created, an energy is spent proportional to the area of the fracture site. We consider as an example that of a brittle elastic bar subject to a forced displacement at its ends, so that volume integrals become line integrals and surface discontinuities turn into jumps. In this case, if $g$ denotes the external body forces acting on the bar, the deformation $u$ of the bar at equilibrium will solve the following problem:

$$
\min \left\{ \int_{(a,b)} f(u') \, dt + \lambda \#(S(u)) - \int_{(a,b)} gu \, dt : u(a) = u_a, \ u(b) = u_b, \ u^+ > u^- \text{ on } S(u) \right\}. \quad (7.5)
$$

on the space of functions $u \in P-W^{1,p}(a,b)$, for some $p > 1$. The function $f$ represents the elastic response of the bar in the unfractured region, while the condition $u^+ > u^-$ derives from the inpenetrability of matter.
We consider energies on $P-W^{1,p}(a,b)$ of the form

$$F(u) = \int_{(a,b)} f(u') \, dt + \sum_{S(u)} \vartheta(u^+ - u^-). \quad (7.6)$$

Lower-semicontinuity and coerciveness properties for such functionals will easily follow from the corresponding properties on $W^{1,p}(a,b)$ and $PC(a,b)$.

**Theorem 7.3** Let $p > 1$.

(i) (Coerciveness) If $(u_j)$ is a sequence in $P-W^{1,p}(a,b)$ such that

$$\sup_j \left( \int_{(a,b)} |u_j'|^p \, dt + \#(S(u_j)) \right) < +\infty \quad (7.7)$$

and for all open sets $I \subset (a,b)$ we have $\liminf_j \inf_I |u_j| < +\infty$, then there exists a subsequence of $(u_j)$ (not relabeled) converging in measure to some $u \in P-W^{1,p}(a,b)$. Moreover, we can write $u_j = v_j + w_j$ with $v_j \in W^{1,p}(a,b)$ and $w_j \in PC(a,b)$, with $v_j$ weakly converging in $W^{1,p}(a,b)$ and $w_j$ converging in measure.

(ii) (Lower semicontinuity) If $f : \mathbb{R} \to [0, +\infty]$ is a convex and lower semicontinuous function, and if $\vartheta : \mathbb{R} \to [0, +\infty]$ is a subadditive and lower semicontinuous function then the functional $F$ defined in (7.6) is lower semicontinuous on $P-W^{1,p}(a,b)$ with respect to convergence in measure along sequences $(u_j)$ satisfying (7.7).

**Proof** (i) Let $v_j \in W^{1,p}(a,b)$ be defined by

$$v_j(t) = \int_a^t u_j'(s) \, ds.$$ 

Since $v_j(a) = 0$ for all $j$, the sequence $(v_j)$ is bounded in $W^{1,p}(a,b)$ by Poincaré inequality, and hence we can extract a weakly converging subsequence (that we still denote by $(v_j)$) that weakly converges to some $v$ in $W^{1,p}(a,b)$. Now, set $w_j = u_j - v_j \in PC(a,b)$. Since $v_j \to v$ in $L^\infty(a,b)$ the sequence $(w_j)$ satisfies the hypotheses of Proposition 2.2, so that, upon extracting a subsequence, it converges in measure to some $w \in PC(a,b)$. The sequence $(u_j)$ satisfies the required properties with $u = v + w$.

(ii) Let $(u_j)$ satisfy (7.7) and $u_j \to u$ in measure. Then by (i) we can write $u_j = v_j + w_j$ with $v_j \in W^{1,p}(a,b)$ and $w_j \in PC(a,b)$, $w_j \to w$ weakly in $W^{1,p}(a,b)$ and $v_j \to v$ in $PC(a,b)$ in measure. By Proposition 4.13 and Theorem 2.7 we then get

$$F(u) = F(v) + F(w) \leq \liminf_j F(v_j) + \liminf_j F(w_j) \leq \liminf_j F(u_j)$$

as desired. □
Corollary 7.4 Let \( f, \vartheta : \mathbb{R} \to [0, +\infty] \) be functions satisfying
\[
c|z|^p \leq f(z) \quad \text{and} \quad c \leq \vartheta(z) \tag{7.8}
\]
for all \( z \in \mathbb{R} \), then the functional \( F \) defined in (7.6) is lower semicontinuous on \( P-W^{1,p}(a,b) \) with respect to convergence in measure if and only if \( f \) is convex and lower semicontinuous and \( \vartheta \) is subadditive and lower semicontinuous.

**Proof** Let \( F \) be lower semicontinuous. Then also its restrictions to \( W^{1,p}(a,b) \) and to \( PC(a,b) \) are lower semicontinuous; hence, we deduce that \( f \) is convex and lower semicontinuous and \( \vartheta \) is subadditive and lower semicontinuous by Proposition 4.11 and Theorem 2.7. The converse is a immediate consequence of Theorem 7.3.

### 7.4 Examples of existence results

As examples of an application of the lower semicontinuity theorems on the space \( P-W^{1,p}(a,b) \) we prove the existence of solutions for the problems outlined in Section 7.2.

**Example 7.5** (Existence in Image Reconstruction problems) We use the notation of Section 7.2.1. Let \( g \in L^2(a,b) \) and let \( (u_j) \) be a minimizing sequence for the problem
\[
m = \inf \left\{ F(u) + c_3 \int_{(a,b)} |u - g|^2 \, dt : u \in P-W^{1,2}(a,b) \right\}, \tag{7.9}
\]
where
\[
F(u) = c_1 \int_{(a,b)} |u'|^2 \, dt + c_2 \#(S(u)).
\]
By taking \( u = 0 \) as a test function, we get that \( m \leq \int_{(a,b)} |g|^2 \, dt \). Moreover, we immediately get that \( (u_j) \) is bounded in \( L^2(a,b) \); hence, it satisfies the hypotheses of Theorem 7.3(i). We can thus suppose that \( u_j \to u \in P-W^{1,2}(a,b) \) in measure and a.e., so that by Theorem 7.3(ii) (with \( p = 2 \), \( f(z) = |z|^2 \) and \( \vartheta(z) = 1 \)) \( F(u) \leq \liminf_j F(u_j) \), and by Fatou’s Lemma
\[
\int_{(a,b)} |u - g|^2 \, dt \leq \liminf_j \int_{(a,b)} |u_j - g|^2 \, dt,
\]
so that \( u \) is a minimum point for (7.9).

**Example 7.6** (Existence in Fracture Mechanics problems) We use the notation of Section 7.2.2. In this case we may have to specify the boundary conditions better, as \( S(u) \) may tend to \( a \) or \( b \); i.e, the elastic bar may break at its ends.
7.4. EXAMPLES OF EXISTENCE RESULTS

In view of Theorem 2.22, the minimization problem with relaxed boundary condition takes the form

\[ m = \inf \left\{ F(u) - \int_{(a,b)} g u \, dt + \vartheta(u(a+) - u_a) + \vartheta(u_b - u(b-)) : u \in P-W^{1,p}(a,b) \right\}, \tag{7.10} \]

where

\[ F(u) = \int_{(a,b)} f(u') \, dt + \sum_{S(u)} \vartheta(u^+ - u^-), \]

\( f \) is some convex function, which we suppose satisfies \( f(z) \geq |z|^p - c \), and \( \vartheta \) is defined by

\[ \vartheta(z) = \begin{cases} +\infty & \text{if } z < 0 \\ 0 & \text{if } z = 0 \\ 1 & \text{if } z > 0. \end{cases} \]

Note that \( \vartheta \) takes care of the inpenetrability condition, which needs not be repeated in the statement of the minimum problem in the form (7.10).

We deal with the case \( u_b > u_a \), and suppose \( f(0) = 0 \) and \( \lambda = 1 \). We may use \( u = (u_a + u_b)/2 \) as a test function, obtaining

\[ m \leq F(u) + \vartheta(u(a+) - u_a) + \vartheta(u_b - u(b-)) = 2 \vartheta \left( \frac{u_b - u_a}{2} \right) = 2. \]

Let \( (u_j) \) be a minimizing sequence for (7.10). We set \( u_j = v_j + w_j \) with \( v_j \in W^{1,p}(a,b) \), \( w_j \in PC(a,b) \) and \( v_j(a) = 0 \). By the Poincaré inequality and the continuous imbedding of \( W^{1,p}(a,b) \) into \( L^\infty(a,b) \) we obtain that

\[ \|v_j\|_{L^\infty(a,b)} \leq c\|v_j'\|_{L^p(a,b)}. \tag{7.11} \]

Note that the condition \( u_j^+ > u_j^- \) implies that \( w_j \) is increasing, so that

\[ \|w_j\|_{L^\infty(a,b)} \leq |u_a| + |u_b| + c\|v_j'\|_{L^p(a,b)}. \tag{7.12} \]

From the condition

\[ \int_{(a,b)} f(u_j') \, dt + \sum_{S(u)} \vartheta(u_j^+ - u_j^-) - \int_{(a,b)} g u_j \, dt \leq c \]

we then get in particular

\[ \int_{(a,b)} |v_j'|^p \, dt - \int_{(a,b)} g v_j \, dt - \int_{(a,b)} g w_j \, dt \leq c, \]

from which we deduce by (7.11)-(7.12)

\[ \int_{(a,b)} |v_j'|^p \, dt - c\|v_j\|_{L^\infty(a,b)} - c\|w_j\|_{L^\infty(a,b)} \leq c \]

and, from the inequalities above, eventually

\[ \int_{(a,b)} |v_j'|^p \, dt \leq c. \]
Hence, we may assume that \( v_j \) weakly converge in \( W^{1,p}(a, b) \), and by (7.12) we obtain that \( (w_j) \) is a bounded sequence in \( L^\infty(a, b) \). Hence \( (u_j) \) satisfies the assumptions of Theorem 7.3(i), so that we may assume that it converges to \( u \) in measure. Moreover, we may assume that \( w_j \) converges a.e. and in \( L^1(a, b) \), so that we get that \( u \) is a minimum point for (7.10) by using Theorem 7.3(ii).

7.5 Exercises

**Exercise 7.1** Let \( u \) describe a brittle elastic bar in a vertical position subject to its weight and fixed at the upper end. Where will it break?