Chapter 6

Limits of non-convex discrete systems

In this chapter we treat the problem of the description of variational limits of discrete problems in a one-dimensional setting. Given \( n \in \mathbb{N} \) and points \( x_n^i = i\lambda_n \) (\( \lambda_n = L/n \) is the lattice spacing) we consider energies of the general form

\[
E_n(\{u_i\}) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \left( \frac{u_{i+j} - u_i}{j\lambda_n} \right).
\]

If we picture the set \( \{x_n^i\} \) as the reference configuration of an array of material points interacting through some forces, and \( u_i \) represents the displacement of the \( i \)-th point, then \( \psi_n^j \) can be thought as the energy density of the interaction of points with distance \( j\lambda_n \) (\( j \) lattice spacings) in the reference lattice. Note that the only assumption we make is that \( \psi_n^j \) depends on \( \{u_i\} \) through the differences \( u_{i+j} - u_i \), but we find it more convenient to highlight its dependence on ‘discrete difference quotients’.

Our goal is to describe the behaviour of problems of the form

\[
\min \left\{ E_n(\{u_i\}) - \sum_{i=0}^{n-1} \lambda_n u_i f_i : u_0 = U_0, \ u_n = U_L \right\}
\]

(and similar), and to show that for a quite general class of energies these problems have a limit continuous counterpart. Here \( \{f_i\} \) represents the external forces and \( U_0, U_L \) are the boundary conditions at the endpoints of the interval \((0, L)\). More general statement and different problems can be also obtained. To make this asymptotic analysis precise, we use the notation and methods of \( \Gamma \)-convergence. We will show that, under some growth conditions, upon suitably identifying discrete functions \( \{u_i\} \) with their interpolations, the free energies \( E_n \) \( \Gamma \)-converge to a limit energy \( F \), which is defined on a Sobolev space and takes
the form
\[ F(u) = \int_0^L \psi(u') \, dt. \]
As a consequence we obtain that minimizers of the problem above are ‘very close’ to minimizers of a classical problem of the Calculus of Variations
\[ \min \left\{ \int_0^L (\psi(u') - fu) \, dt : u(0) = U_0, \; u(L) = U_L \right\}. \]
Even though the description of this limit passage can be performed in a much more general setting, for the time being, we will treat only the case when the limit is defined in a Sobolev space, and either we have only nearest-neighbour interactions: we can write
\[ E_n(\{u_i\}) = \sum_{i=0}^{n-1} \lambda_n \psi_n \left( \frac{u_{i+1} - u_i}{\lambda_n} \right), \]
or up to next-to-nearest neighbours; i.e., when \( \psi_n^j = 0 \) if \( j > 2 \).

### 6.1 Discrete functionals

We will consider the limit of energies defined on one-dimensional discrete systems of \( n \) points as \( n \) tends to \( +\infty \). In order to define a limit energy on a continuum we parameterize these points as a subset of a single interval \((0, L)\). Set
\[ \lambda_n = \frac{L}{n}, \quad x_i^n = \frac{i}{n} L = i \lambda_n, \quad i = 0, 1, \ldots, n. \quad (6.1) \]
We denote \( I_n = \{x_0^n, \ldots, x_n^n\} \) and by \( \mathcal{A}_n(0, L) \) the set of functions \( u : I_n \to \mathbb{R} \). If \( n \) is fixed and \( u \in \mathcal{A}_n(0, L) \) we equivalently denote
\[ u_i = u(x_i^n). \]
Given \( K \in \mathbb{N} \) with \( 1 \leq K \leq n \) and functions \( f^j : \mathbb{R} \to [0, +\infty) \), with \( j = 1, \ldots, K \), we will consider the related functional \( E : \mathcal{A}_n(0, L) \to [0, +\infty] \) given by
\[ E(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} f^j(u_{i+j} - u_i). \quad (6.2) \]
Note that \( E \) can be viewed simply as a function \( E : \mathbb{R}^n \to [0, +\infty] \).

An interpretation with a physical flavour of the energy \( E \) is as the internal interaction energy of a chain of \( n + 1 \) material points each one interacting with its \( K \)-nearest neighbours, under the assumption that the interaction energy densities depend only on the order \( j \) of the interaction and on the distance between the two points \( u_{i+j} - u_i \) in the reference configuration. If \( K = 1 \) then each point interacts with its nearest neighbour only, while if \( K = n \) then each pair of points interacts.
6.1. DISCRETE FUNCTIONALS

**Remark 6.1** From elementary calculus we have that $E$ is lower semicontinuous if each $f^j$ is lower semicontinuous, and that $E$ is coercive on bounded sets of $A_n(0, L)$.

We will study the limit as $n \to +\infty$ of sequences $(E_n)$ with $E_n : A_n(0, L) \to [0, +\infty]$ of the general form

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} f^j_n(u_{i+j} - u_i),$$  \hspace{1cm} (6.3)

and show that it provides an energy defined on a Sobolev space.

Since each functional $E_n$ is defined on a different space, the first step is to identify each $A_n(0, L)$ with a subspace of a common space of functions defined on $(0, L)$. In order to identify each discrete function with a continuous counterpart, we extend $u$ by $\tilde{u} : (0, L) \to \mathbb{R}$ as the piecewise-affine function defined by

$$\tilde{u}(s) = u_{i-1} + \frac{u_i - u_{i-1}}{\lambda_n}(s - x_{i-1}) \quad \text{if } s \in (x_{i-1}, x_i).$$  \hspace{1cm} (6.4)

In this case, $A_n(0, L)$ is identified with those continuous $u \in W^{1,1}(0, L)$ (actually, in $W^{1,\infty}(0, L)$) such that $u$ is affine on each interval $(x_{i-1}, x_i)$. Note moreover that we have

$$\tilde{u}' = \frac{u_i - u_{i-1}}{\lambda_n}$$  \hspace{1cm} (6.5)

on $(x_{i-1}, x_i)$. If no confusion is possible, we will simply write $u$ in place of $\tilde{u}$.

As we will treat limit functionals defined on Sobolev spaces, it is convenient to rewrite the dependence of the energy densities in (6.3) with respect to difference quotients rather than the differences $u_{i+j} - u_i$. We then write

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} \lambda_n \psi^j_n \left( \frac{u_{i+j} - u_i}{\lambda_n} \right),$$  \hspace{1cm} (6.6)

where

$$\psi^j_n(z) = \frac{1}{\lambda_n} f^j_n(z).$$

With the identification of $u$ with $\tilde{u}$, $E_n$ may be viewed as an integral functional defined on $W^{1,1}(0, L)$ by the equality

$$E_n(u) = F_n(\tilde{u}),$$  \hspace{1cm} (6.7)

where

$$F_n(v) = \begin{cases} \sum_{j=1}^{K_n} \sum_{i=0}^{j-1} \int_{x_{i+1}}^{x_{i+j+1}} \psi^j_n \left( \frac{1}{\lambda_n} \sum_{k=-l}^{j-1} v'(x + k\lambda_n) \right) dx & \text{if } v \in A_n(0, L) \\ +\infty & \text{otherwise.} \end{cases}$$  \hspace{1cm} (6.8)
Note that in the particular case $K_n = 1$ we have (set $\psi_n = \psi_n^1$)

$$F_n(v) = \begin{cases} 
\int_0^L \psi_n(v')dx & \text{if } v \in \mathcal{A}_n(0, L) \\
+\infty & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (6.9)

**Definition 6.2 (Convergence of discrete functions and energies)** With the identifications above we will say that $u_n \text{ converges to } u$ (respectively, in $L^1$, in measure, in $W^{1,1}$, etc.) if $\tilde{u}_n$ converge to $u$ (respectively, in $L^1$, in measure, weakly in $W^{1,1}$, etc.), and we will say that $E_n \Gamma \text{-converges to } F$ (respectively, with respect to the convergence in $L^1$, in measure, weakly in $W^{1,1}$, etc.) if $F_n \Gamma \text{-converges to } F$ (respectively, with respect to the convergence in $L^1$, in measure, weakly in $W^{1,1}$, etc.); i.e., if for all $u$

(i) (liminf inequality) $F(u) \leq \liminf_n F_n(u_n)$ for all $u_n$ converging to $u$;

(ii) (limsup inequality) there exists $u_n$ converging to $u$ such that $\limsup_n F_n(u) \leq F(u)$.

We recall that since our functionals will always be equi-coercive then $\Gamma$-convergence entails the desired convergence of minimum problems as described in the previous section.

### 6.2 Convex energies

First we briefly recall the case when the energies $\psi_n^i$ are convex. We will see that in the case of nearest neighbours, the limit is obtained by simply replacing sums by integrals, while in the case of long-range interactions a superposition principle holds.

#### 6.2.1 Nearest-neighbour interactions: an identification principle

We start by considering the case $K = 1$, so that the functionals $E_n$ are given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_n \left( \frac{u_{i+1} - u_i}{\lambda_n} \right).$$  \hspace{1cm} (6.10)

The integral counterpart of $E_n$ is given by

$$F_n(v) = \begin{cases} 
\int_0^L \psi_n(v')dx & \text{if } v \in \mathcal{A}_n(0, L) \\
+\infty & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (6.11)

The following result states that as $n$ approaches $\infty$ the identification of $E_n$ with its continuous analog is complete.
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**Theorem 6.3** Let \( \psi_n : \mathbb{R} \to [0, +\infty) \) be convex and locally equi-bounded. Let \( E_n \) be given by (6.10) and let \( \psi = \lim_n \psi_n \) (note that it is not restrictive to suppose that such limit exists upon extraction a subsequence).

(i) The \( \Gamma \)-limit of \( E_n \) with respect to the weak convergence in \( W^{1,1}(0, L) \) is given by \( F \) defined by

\[
F(u) = \int_{(0,L)} \psi(u') \, dx. \tag{6.12}
\]

(ii) If

\[
\lim_{|z| \to \infty} \frac{\psi(z)}{|z|} = +\infty \tag{6.13}
\]

then the \( \Gamma \)-limit of \( E_n \) with respect to the convergence in \( L^1(0,L) \) is given by \( F \) defined by

\[
F(u) = \begin{cases} 
\int_{(0,L)} \psi(u') \, dx & \text{if } u \in W^{1,1}(0,L) \\
+\infty & \text{otherwise}
\end{cases} \tag{6.14}
\]
on \( L^1(0,L) \).

**Proof** (i) We have \( \Gamma \)-\( \liminf_j F_j(u) \geq F(u) \). Conversely, fixed \( u \in W^{1,\infty}(0,L) \) let \( u_n \in \mathcal{A}_n(0,L) \) be such that \( u_n(x_i^n) = u(x_i^n) \). By convexity we have

\[
\int_{x_i^n}^{x_{i+1}^n} \psi(u') \, dt \geq \lambda_n \psi \left( \frac{1}{\lambda_n} \int_{x_i^n}^{x_{i+1}^n} u' \, dt \right) = \lambda_n \psi \left( \frac{u(x_{i+1}^n) - u(x_i^n)}{\lambda_n} \right),
\]
hence, summing up, letting \( n \to +\infty \) and using the pointwise convergence of \( \psi_n \) to \( \psi \), we get

\[
\int_0^L \psi(u') \, dt = \lim_n \int_0^L \psi_n(u') \, dt \geq \limsup_n E_n(u_n).
\]

By a density argument we recover the same inequality on the whole \( W^{1,1}(0,L) \).

(ii) If (6.13) holds then the sequence \( (E_n) \) is equi-coercive on bounded sets of \( L^1(0,L) \) with respect to the weak convergence in \( W^{1,1}(0,L) \), from which the thesis is easily deduced. \( \square \)

6.2.2 Long-range interactions: a superposition principle

Let now \( K \in \mathbb{N} \) be fixed. The energies \( E_n \) take the form

\[
E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n \left( \frac{u_{i+j} - u_i}{j \lambda_n} \right). \tag{6.15}
\]

In this case \( E_n \) can be seen as the superposition of \( K \) nearest-neighbour functionals to which we can apply the result above.
Theorem 6.4 Let \( \psi_n : \mathbb{R} \to [0, +\infty) \) be convex and locally equi-bounded. Let \( E_n \) be given by (6.15) and for all \( j \) let \( \psi_j = \lim_n \psi_n \) (note that it is not restrictive to suppose that such limit exists upon extraction a subsequence). Let \( \psi^j \) satisfy
\[
\lim_{|z| \to \infty} \frac{\psi^j(z)}{|z|} = +\infty
\]
then the \( \Gamma \)-limit of \( E_n \) with respect to the convergence in \( L^1(0,L) \) is given by \( F \) defined by
\[
F(u) = \begin{cases} 
\int_{(0,L)} \psi(u') \, dx & \text{if } u \in W^{1,1}(0,L) \\
+\infty & \text{otherwise}
\end{cases}
\]
on \( L^1(0,L) \), where
\[
\psi = \sum_{j=1}^K \psi^j.
\]
\[ \square \]

6.3 Non-convex energies

We now investigate the effects of the lack of convexity.

6.3.1 Nearest-neighbour interactions: a convexification principle

In the case \( K = 1 \) the only effect of the passage from the discrete setting to the continuum is a convexification of the integrand.

Theorem 6.5 Let \( \psi_n : \mathbb{R} \to [0, +\infty) \) be locally equi-bounded Borel functions satisfying (6.13), and suppose that \( \psi = \lim_n \psi_n^{**} \). Let \( E_n \) be given by (6.10); then the \( \Gamma \)-limit of \( E_n \) with respect to the convergence in \( L^1(0,L) \) is given by \( F \) defined by
\[
F(u) = \begin{cases} 
\int_{(0,L)} \psi(u') \, dx & \text{if } u \in W^{1,1}(0,L) \\
+\infty & \text{otherwise}
\end{cases}
\]
on \( L^1(0,L) \). In particular if \( \psi_n = \tilde{\psi} \) then \( \psi = \tilde{\psi}^{**} \).

\[ \square \]
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6.3.2 Next-to-nearest neighbour interactions: non-convex relaxation

In the non-convex setting, the case \( K = 2 \) offers an interesting way of describing the two-level interactions between first and second neighbours. Such description is more difficult in the case \( K \geq 3 \). Essentially, the way the limit continuum theory is obtained is by first integrating out the contribution due to nearest neighbours by means of an inf-convolution procedure and then by applying the previous results to the resulting functional.

**Theorem 6.6** Let \( \psi_n^1, \psi_n^2 : \mathbb{R} \to [0, +\infty) \) be locally equi-bounded Borel functions such that
\[
\lim_{|z| \to \infty} \frac{\psi_n^1(z)}{|z|} = +\infty,
\]
uniformly in \( n \), and let \( E_n(u) : \mathcal{A}_n(0, L) \to [0, +\infty) \) be given by
\[
E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_n^1 \left( \frac{u_{i+1} - u_i}{\lambda_n} \right) + \sum_{i=0}^{n-2} \lambda_n \psi_n^2 \left( \frac{u_{i+2} - u_i}{2\lambda_n} \right)
\]
(6.21)
Let \( \tilde{\psi}_n : \mathbb{R} \to [0, +\infty) \) be defined by
\[
\tilde{\psi}_n(z) = \psi_n^2(z) + \frac{1}{2} \inf \{ \psi_n^1(z_1) + \psi_n^1(z_2) : z_1 + z_2 = 2z \}
\]
\[
= \inf \left\{ \psi_n^2(z) + \frac{1}{2} (\psi_n^1(z_1) + \psi_n^1(z_2)) : z_1 + z_2 = 2z \right\},
\]
(6.22)
and suppose that
\[
\psi = \lim_{n \to \infty} \tilde{\psi}_n^*.
\]
(6.23)
Then the \( \Gamma \)-limit of \( E_n \) with respect to the convergence in \( L^1(0, L) \) is given by \( F \) defined by
\[
F(u) = \begin{cases} 
\int_{(0, L)} \psi(u') \, dx & \text{if } u \in W^{1,1}(0, L) \\
+\infty & \text{otherwise}
\end{cases}
\]
(6.24)
on \( L^1(0, L) \).

**Remark 6.7** (i) The growth conditions on \( \psi_n^2 \) can be weakened, by requiring that \( \psi_n^2 : \mathbb{R} \to \mathbb{R} \) and
\[-c_1 \psi_n^1 \leq \psi_n^2 \leq c_2 (1 + \psi_n^1),
\]
provided that we still have
\[
\lim_{|z| \to \infty} \frac{\psi(z)}{|z|} = +\infty.
\]
(ii) If \( \psi_n^1 \) is convex then \( \tilde{\psi}_n = \psi_n^1 + \psi_n^2 \). If also \( \psi_n^2 \) is convex then we recover a particular case of Theorem 6.4.
Proof Let \( u \in A_n(0, L) \). We have, regrouping the terms in the summation,

\[
E_n(u) = \sum_{i=0}^{n-2} \lambda_n \left( \psi_n^2 \left( \frac{u_{i+2} - u_i}{2 \lambda_n} \right) + \frac{1}{2} \psi_n^1 \left( \frac{u_{i+2} - u_{i+1}}{\lambda_n} \right) + \frac{1}{2} \psi_n^1 \left( \frac{u_{i+1} - u_i}{\lambda_n} \right) \right) \\
+ \sum_{i=0}^{n-2} \lambda_n \left( \psi_n^2 \left( \frac{u_{i+2} - u_i}{2 \lambda_n} \right) + \frac{1}{2} \psi_n^1 \left( \frac{u_{i+2} - u_{i+1}}{\lambda_n} \right) + \frac{1}{2} \psi_n^1 \left( \frac{u_{i+1} - u_i}{\lambda_n} \right) \right) \\
+ \frac{\lambda_n}{2} \psi_n^1 \left( \frac{u_n - u_{n-1}}{\lambda_n} \right) + \frac{1}{2} \psi_n^1 \left( \frac{u_1 - u_0}{\lambda_n} \right) \\
\geq \frac{1}{2} \left( \sum_{i=0}^{n-2} 2 \lambda_n \psi_n \left( \frac{u_{i+2} - u_i}{2 \lambda_n} \right) + \sum_{i=0}^{n-2} 2 \lambda_n \psi_n \left( \frac{u_{i+2} - u_i}{2 \lambda_n} \right) \right) \\
\geq \frac{1}{2} \left( \sum_{i=0}^{n-2} 2 \lambda_n \psi_n^* \left( \frac{u_{i+2} - u_i}{2 \lambda_n} \right) + \sum_{i=0}^{n-2} 2 \lambda_n \psi_n^* \left( \frac{u_{i+2} - u_i}{2 \lambda_n} \right) \right) \\
= \frac{1}{2} \left( \int_0^{2 \lambda_n \lfloor n/2 \rfloor} \psi_n^* \left( \tilde{u}_1 \right) dt + \int_{\lambda_n}^{\left( 1 + 2 \lfloor n/2 \rfloor \right) \lambda_n} \psi_n^* \left( \tilde{u}_2 \right) dt \right), \tag{6.25} \end{align*}

where \( \tilde{u}_k \), respectively, with \( k = 1, 2 \), are the continuous piecewise-affine functions such that

\[
\tilde{u}_k' = \frac{u_{i+2} - u_i}{2 \lambda_n} \quad \text{on} \quad (x_i^n, x_{i+2}^n) \tag{6.26} \]

for \( i \), respectively, even or odd.

Let now \( u_n \rightharpoonup u \) in \( L^1(0, L) \) and \( \sup_n E_n(u_n) < +\infty \); then \( u_n \rightharpoonup u \) in \( W^{1,1}(0, L) \). Let \( u_{k,n} \) be defined as in (6.26); we then deduce that \( u_{k,n} \rightharpoonup u \) as \( n \to +\infty \), for \( k = 1, 2 \). For every fixed \( \eta > 0 \) by (6.25) we obtain

\[
\liminf_n E_n(u_n) \geq \frac{1}{2} \left( \liminf_n \int_{\eta}^{\eta} \psi_n^* \left( u_{1,n} \right) dt + \liminf_n \int_{\eta}^{\eta} \psi_n^* \left( u_{2,n} \right) dt \right) \\
\geq \int_{\eta}^{\eta} \psi(u') dt,
\]

and the liminf inequality follows by the arbitrariness of \( \eta > 0 \).

Now we prove the limsup inequality. By an easy relaxation argument, it suffices to treat the case when \( \psi_n \) is lower semicontinuous, \( u(x) = z x \) and \( \psi(z) = \lim_n \tilde{\psi}_n(z) \). With fixed \( \eta > 0 \) let \( z_1^n, z_2^n \) be such that \( z_1^n + z_2^n = 2z \) and

\[
\psi_n^2(z) + \frac{1}{2} (\psi_n^1(z_1^n) + \psi_n^1(z_2^n))^2 \leq \tilde{\psi}(z) + \eta
\]

for all \( n \) sufficiently large. We define the recovery sequence \( u_n \) as

\[
u_n(x^n_i) = \begin{cases} 
  z x^n_i & \text{if } i \text{ is even} \\
  z (i-1) \lambda_n + z^n_i \lambda_n & \text{if } i \text{ is odd}
\end{cases}
\]
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We then have
\[ E_n(u_n) = \sum_{i=0}^{n-1} \lambda_n \psi_1^2 \left( \frac{u_n(x^n_{i+1}) - u_n(x^n_i)}{\lambda_n} + \sum_{i=0}^{n-2} \lambda_n \psi_2^2 \left( \frac{u_n(x^n_{i+2}) - u_n(x^n_{i+1})}{2\lambda_n} \right) \right) \leq \frac{L}{2} \left( \psi_1(z^n_1) + \psi_1(z^n_2) \right) + L \lambda_n^2 \psi(z) \leq L \psi(z) + L \eta, \]
and the limsup inequality follows by the arbitrariness of \( \eta \).

\[ \square \]

Remark 6.8 (Multiple-scale effects) The formula defining \( \psi \) highlights a double-scale effect. The operation of inf-convolution highlights oscillations on the scale \( \lambda_n \), while the convexification of \( \hat{\psi} \) acts at a much larger scale.

6.3.3 Long-range interactions: a ‘clustering’ principle

We consider now the case of a general \( K \geq 1 \). In this case the effective energy density will be given by a homogenization formula. We just state and comment the general result, without including the proof.

Theorem 6.9 Let \( K \geq 1 \). Let \( \psi_j : \mathbb{R} \to [0, +\infty) \) be lower semicontinuous functions and let \( p > 1 \) exists such that
\[ \psi_j^{\prime}(z) \geq c_0 (|z|^p - 1), \quad \psi_j^{\prime}(z) \leq c_j (1 + |z|^p). \quad (6.27) \]
for all \( j \in \{1, \ldots, K\} \) and \( n \in \mathbb{N} \). For all \( N, n \in \mathbb{N} \) let \( \psi_{N,n} : \mathbb{R} \to [0, +\infty) \) be defined by
\[ \psi_{N,n}(z) = \min \left\{ \frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi_j \left( \frac{u(i+j) - u(i)}{j} \right) \right\} \]
\[ u : \{0, \ldots, N\} \to \mathbb{R}, \quad u(i) = zi \text{ for } i \leq K \text{ or } i \geq N - K \quad (6.28) \]
Suppose that \( \psi : \mathbb{R} \to [0, +\infty) \) exists such that
\[ \psi(z) = \lim_n \lim_{N} \psi_{N,n}(z) \quad \text{for all } z \in \mathbb{R} \quad (6.29) \]
(note that this is not restrictive upon passing to a subsequence of \( n \) and \( N \)). Let \( E_n \) be defined on \( A_n(0, L) \) by
\[ E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_j^2 \left( \frac{u_{i+j} - u_i}{j\lambda_n} \right). \quad (6.30) \]
Then the \( \Gamma \)-limit of \( E_n \) with respect to the convergence in \( L^1(0, L) \) is given by \( F \) defined by
\[ F(u) = \begin{cases} \int_{(0, L)} \psi(u') \, dx & \text{if } u \in W^{1,p}(0, L) \\ +\infty & \text{otherwise} \end{cases} \quad (6.31) \]
on \( L^1(0, L) \).
Remark 6.10 (Clustering) The main idea of the result above is to show that (upon some controllable errors) we can find a lattice spacing $\eta_n$ (possibly much larger than $\lambda_n$) such that $E_n$ is ‘equivalent’ (as $\Gamma$-convergence is concerned) to a nearest-neighbour interaction energy on a lattice of step size $\eta_n$, of the form

$$E_n(\{u_i\}) = \sum_{i=0}^{m-1} \eta_n \psi_n \left( \frac{u_{i+1} - u_i}{\eta_n} \right),$$

and to which then the previous results can be applied.

In this case the reasoning that leads from $E_n$ to $\overline{E}_n$ is that the overall behaviour of a system of interacting point will behave as clusters of large arrays of neighbouring points interacting through their ‘extremities’.

### 6.3.4 Convergence of minimum problems

From Theorem 6.9 we immediately deduce the following theorem.

**Theorem 6.11** Let $E_n$ and $F$ be given by Theorem 6.9, let $f \in L^1(0, L)$ and $d > 0$. Then the minimum values

$$m_n = \min \left\{ E_n(u) + \int_0^L f u \, dt : \ u(0) = 0, \ u(L) = d \right\}$$

converge to

$$m = \min \left\{ F(u) + \int_0^L f u \, dt : \ u(0) = 0, \ u(L) = d \right\},$$

and from each sequence of minimizers of (6.32) we can extract a subsequence converging to a minimizer of (6.33).

**Proof** Since the sequence of functionals $(E_n)$ is equi-coercive, it suffices to show that the boundary conditions do not change the form of the $\Gamma$-limit; i.e., that for all $u \in W^{1,p}(0, L)$ such that $u(0) = 0$ and $u(L) = d$ and for all $\varepsilon > 0$ there exists a sequence $u_n$ such that $u_n(0) = 0$, $u_n(L) = d$ and $\limsup_n E_n(u_n) \leq F(u) + \varepsilon$.

Let $v_n \to u$ in $L^\infty(0, L)$ be such that $\lim_n E_n(v_n) = F(u)$. With fixed $\eta > 0$ and $N \in \mathbb{N}$ let $K_n \in \mathbb{N}$ be such that

$$\lim_n K_n \lambda_n = \frac{\eta}{N}.$$ 

For all $l \in \{1, \ldots, N\}$ let $\phi_n^{N,l} : [0, L] \to [0, 1]$ be the piecewise-affine function defined by $\phi_n^{N,l}(0) = 0$, 

$$\phi_n^{N,l} = \begin{cases} 1/(K_n \lambda_n) & \text{on } ((l-1)K_n \lambda_n, lK_n \lambda_n) \\ -1/(K_n \lambda_n) & \text{on } ((n-lK_n) \lambda_n, (n-lK_n + K_n) \lambda_n) \\ 0 & \text{otherwise.} \end{cases}$$


Let
\[ u_{n}^{N,\ell} = \phi_{n}^{N,\ell} v_{n} + (1 - \phi_{n}^{N,\ell}) u. \]

We have
\[
E_{n}(u_{n}^{N,\ell}) \leq E_{n}(u_{n}) + c \left( \int_{0}^{\eta + \lambda_{n}} (1 + |u'|^{p}) \, dt + \int_{L - \eta - \lambda_{n}}^{L} (1 + |u'|^{p}) \, dt \right) \\
+ c \left( \int_{(l-1)K_{n} - K \lambda_{n} - (lK_{n} + K \lambda_{n}) \cap (0, L)} |u_{n}^{\ell}'|^{p} \, dt \right) \\
+ \int_{(n-lK_{n} - K \lambda_{n} - (n-lK_{n} + K \lambda_{n}) \cap (0, L)} |v_{n}|^{p} \, dt \\
+ \int_{0}^{L} \frac{1}{(K_{n} \lambda_{n})^{p}} |v_{n} - u|^{p} \right) \\
\leq E_{n}(u_{n}) + c \left( \int_{0}^{2\eta} (1 + |u'|^{p}) \, dt + \int_{L-2\eta}^{L} (1 + |u'|^{p}) \, dt \right) \\
+ c \left( \int_{((l-2)\eta/N, ((l+1)\eta/N) \cup (L-(l+1)\eta/N, L-(l-2)\eta/N)) \cap (0, L)} |v_{n}'|^{p} \, dt \right) \\
+ c \frac{N^{p}}{\eta^{p}} \|v_{n} - u\|_{L^{\infty}(0, L)}^{p}
\]
for \( n \) large enough. Since
\[
\sum_{l=1}^{N} \int_{((l-2)\eta/N, ((l+1)\eta/N) \cup (L-(l+1)\eta/N, L-(l-2)\eta/N) \cap (0, L)} |v_{n}'|^{p} \, dt
\]
\[
\leq 2 \int_{0}^{L} (1 + |v_{n}'|^{p}) \, dt \leq c,
\]
for all \( n \) there exists \( l_{n} \in \{1, \ldots, N\} \) such that
\[
E_{n}(u_{n}^{N,\ell_{n}}) \leq E_{n}(v_{n}) + c \left( \int_{0}^{2\eta} (1 + |u'|^{p}) \, dt + \int_{L-2\eta}^{L} (1 + |u'|^{p}) \, dt \right) \\
+ \frac{c}{N} + c \frac{N^{p}}{\eta^{p}} \|v_{n} - u\|_{L^{\infty}(0, L)}^{p}
\]
Setting \( u_{n} = u_{n}^{N,\ell_{n}} \) we then have
\[
\limsup_{n} E_{n}(u_{n}) \leq F(u) + c \left( \int_{0}^{2\eta} (1 + |u'|^{p}) \, dt + \int_{L-2\eta}^{L} (1 + |u'|^{p}) \, dt \right) + \frac{c}{N},
\]
and the desired inequality by the arbitrariness of \( \eta \) and \( N \). \qed

### 6.4 Exercises

**Exercise 6.1** Compute the limit in Theorem 6.6 when \( \psi_{n}^{1}(z) = \psi^{1}(z) = 2(|z| - 1)^{2} \) and \( \psi_{n}^{2}(z) = \psi^{2}(z) = z^{2} \).
Exercise 6.2 Compute the limit in Theorem 6.6 when $\psi_1^1(z) = \psi^1(z) = 2(|z| - 1)^2$ and $\psi_2^1(z) = \psi^2(z) = -z^2$. 